Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

European Journal of Combinatorics 27 (2006) 513–517

European Journal of Combinatorics

www.elsevier.com/locate/ejc

Cohen–Macaulay polymatroidal ideals☆

Jürgen Herzog[a], Takayuki Hibi[b]

[a] *Fachbereich Mathematik und Informatik, Universität Duisburg-Essen, Campus Essen, 45117 Essen, Germany*

[b] *Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan*

Received 7 September 2004; accepted 10 January 2005

Available online 1 February 2005

Abstract

All Cohen–Macaulay polymatroidal ideals are classified. The Cohen–Macaulay polymatroidal ideals are precisely the principal ideals, the Veronese ideals, and the square-free Veronese ideals.

MSC: 13C14; 05B35

1. Introduction

Our goal is to classify all Cohen–Macaulay polymatroid ideals. It can be expected that such classification will be possible, because it seems likely that Cohen–Macaulay monomial ideals with linear resolutions will be quite rare and it is known that every polymatroid ideal has a linear resolution. In general, however, to classify all Cohen–Macaulay monomial ideals with linear resolutions seems rather difficult, because the two algebraic properties depend on the base field.

Our main result says that a polymatroidal ideal I is Cohen–Macaulay if and only if I is a principal ideal, a Veronese ideal, or a square-free Veronese ideal; see Theorem 4.2.

☆ This paper was completed while the authors stayed at the Mathematisches Forschungsinstitut in Oberwolfach in the framework of the Research in Pairs Program.

E-mail addresses: juergen.herzog@uni-essen.de (J. Herzog), hibi@math.sci.osaka-u.ac.jp (T. Hibi).

0195-6698/$ - see front matter
doi:10.1016/j.ejc.2005.01.004

2. Monomial ideals with linear quotients

Let K be a field and $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K with each $\deg x_i = 1$. Let $I \subset S$ be a monomial ideal and $G(I)$ be its unique minimal monomial generators.

A *vertex cover* of I is a subset W of $\{x_1, \ldots, x_n\}$ such that each $u \in G(I)$ is divided by some $x_i \in W$. Such a vertex cover W is called *minimal* if no proper subset of W is a vertex cover of I.

A monomial ideal is called *unmixed* if all minimal vertex covers of I have the same cardinality. If I is *Cohen–Macaulay*, i.e., the quotient ring S/I is Cohen–Macaulay, then I is unmixed. Let $h(I)$ denote the minimal cardinality of the vertex covers of I. It then follows that

$$\dim S/I = n - h(I). \tag{1}$$

We say that a monomial ideal $I \subset S$ has *linear quotients* if there is an ordering $u_1, \ldots, u_s$ of the monomials belonging to $G(I)$ with $\deg u_1 \leq \deg u_2 \leq \cdots \leq \deg u_s$ such that, for each $2 \leq j \leq s$, the colon ideal $(u_1, u_2, \ldots, u_{j-1}) : u_j$ is generated by a subset of $\{x_1, \ldots, x_n\}$.

It is known, e.g., [1, Lemma 4.1], that if a monomial ideal I generated in one degree has linear quotients, then I has a linear resolution.

Let I be a monomial ideal with linear quotient with respect to the ordering $u_1, \ldots, u_s$ of the monomials belonging to $G(I)$. We write $q_j(I)$ for the number of variables which is required to generate the colon ideal $(u_1, u_2, \ldots, u_{j-1}) : u_j$. Let $q(I) = \max_{2 \leq j \leq s} q_j(I)$. It is proved [3, Corollary 1.6] that the length of the minimal free resolution of S/I over S is equal to $q(I) + 1$. Hence

$$\operatorname{depth} S/I = n - q(I) - 1. \tag{2}$$

Thus in particular the integer $q(I)$ is independent of the particular choice of the ordering of the monomials which gives linear quotients.

By using the formulae (1) and (2), it follows that a monomial ideal I with linear quotients is Cohen–Macaulay if and only if $h(I) = q(I) + 1$.

3. Review on polymatroidal ideals

One of the important classes of monomial ideals with linear quotients is the class of polymatroid ideals.

Let, as before, K be a field and $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K with each $\deg x_i = 1$. Let $I \subset S$ be a monomial ideal generated in one degree. We say that I is *polymatroidal* if the following "exchange condition" is satisfied: For monomials $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ belonging to $G(I)$ and for each i with $a_i > b_i$, one has j with $a_j < b_j$ such that $x_j u / x_i \in G(I)$. The reason that we call such an ideal polymatroidal is that the monomials of the ideal correspond to the bases of a discrete polymatroid [2]. The polymatroidal ideal I is called *matroidal* if I is generated by square-free monomials.

The exchange property for polymatroidal ideals has a "dual version" stated below.

Lemma 3.1. *Let I be a polymatroidal ideal. Then, for monomials $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ belonging to $G(I)$ and for each i with $a_i < b_i$, one has j with $a_j > b_j$ such that $x_i u / x_j \in G(I)$.*

Proof. We introduce the distance of u and v by setting $\mathrm{dist}(u, v) = \frac{1}{2} \sum_{q=1}^{n} |a_q - b_q|$. Fix i with $a_i < b_i$. If there is $k_1 \neq i$ with $a_{k_1} < b_{k_1}$, then there is ℓ_1 with $a_{\ell_1} > b_{\ell_1}$ such that $w_1 = x_{\ell_1} v / x_{k_1} \in G(I)$. Let $w_1 = x_1^{c_1} \cdots x_n^{c_n}$. Then $c_i = b_i$ and $\mathrm{dist}(u, w_1) < \mathrm{dist}(u, v)$. Again, if there is $k_2 \neq i$ with $a_{k_2} < c_{k_2}$, then there is ℓ_2 with $a_{\ell_2} > c_{\ell_2}$ such that $w_2 = x_{\ell_2} w_1 / x_{k_2} \in G(I)$. Let $w_2 = x_1^{d_1} \cdots x_n^{d_n}$. Then $d_i = b_i$ and $\mathrm{dist}(u, w_2) < \mathrm{dist}(u, w_1)$. Repeating these procedures yields $w^* = x_1^{q_1} \cdots x_n^{q_n} \in G(I)$ with $q_i = b_i > a_i$ and $q_j \leq a_j$ for all $j \neq i$. One has $j_0 \neq i$ with $q_{j_0} < a_{j_0}$. Then $x_i u / x_{j_0} \in G(I)$, as desired. □

It is known [1, Theorem 5.2] that a polymatroidal ideal has linear quotients with respect to the reverse lexicographic order $<_{rev}$ induced by any ordering $x_{i_1} > x_{i_2} > \cdots > x_{i_n}$. More precisely, if I is a polymatroidal ideal and if $u_1, \ldots, u_s$ are the monomials belonging to $G(I)$ ordered by the reverse lexicographic order, i.e., $u_s <_{rev} \cdots <_{rev} u_2 <_{rev} u_1$, then the colon ideal $(u_1, \ldots, u_{j-1}) : u_j$ is generated by a subset of $\{x_1, \ldots, x_n\}$.

The product of polymatroidal ideals is again polymatroidal [1,2]. In particular each power of a polymatroidal ideal is polymatroidal.

We close the present section with polymatroidal ideals of special kinds which are of great interest to us.

Example 3.2. (a) The *Veronese ideal* of degree d in the variables $x_{i_1}, \ldots, x_{i_t}$ is the dth power $(x_{i_1}, \ldots, x_{i_t})^d$ of the polymatroidal ideal $(x_{i_1}, \ldots, x_{i_t}) \subset S$. The Veronese ideal is polymatroidal and is Cohen–Macaulay.

(b) The *square-free Veronese ideal* of degree d in the variables $x_{i_1}, \ldots, x_{i_t}$ is the ideal of S which is generated by all square-free monomials in $x_{i_1}, \ldots, x_{i_t}$ of degree d. The square-free Veronese ideal is matroidal and is Cohen–Macaulay.

To see why the square-free Veronese ideal is Cohen–Macaulay, let I denote the square-free Veronese ideal of degree d in the variables $x_1, \ldots, x_n$. Then I coincides with the Stanley–Reisner ideal $I_\Delta \subset S$ of the simplicial complex Δ on the vertex set $\{1, \ldots, n\}$ whose facets are the $(d-1)$-element subsets of $\{1, \ldots, n\}$. Since the simplicial complex Δ is a skeleton of the $(n-1)$-simplex, it follows that Δ is Cohen–Macaulay.

4. Classification of Cohen–Macaulay polymatroidal ideals

We now classify all Cohen–Macaulay polymatroidal ideals. Recall that the support of a monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ is $\mathrm{supp}(u) = \{x_i : a_i \neq 0\}$.

Lemma 4.1. *If $I \subset S$ is a Cohen–Macaulay polymatroidal ideal, then its radical $\sqrt{I}$ is square-free Veronese.*

Proof. Let $I \subset S$ be a Cohen–Macaulay polymatroidal ideal. We may assume that $\bigcup_{u \in G(I)} \mathrm{supp}(u) = \{x_1, \ldots, x_n\}$. Let $u \in G(I)$ be a monomial for which $|\mathrm{supp}(u)|$ is minimal. Let, say, $\mathrm{supp}(u) = \{x_{n-d+1}, x_{n-d+2}, \ldots, x_n\}$. Let J denote the monomial ideal generated by those monomials $w \in G(I)$ such that w is bigger than u with respect to the reverse lexicographic order induced by the ordering $x_1 > x_2 > \cdots > x_n$. We know that the colon ideal $J : u$ is generated by a subset M of $\{x_1, \ldots, x_n\}$. We claim that $\{x_1, \ldots, x_{n-d}\} \subset M$. For each $1 \leq i \leq n-d$, there is a monomial belonging to $G(I)$ which is divided by x_i. It follows from Lemma 3.1 that there is a variable x_j with $n-d+1 \leq j \leq n$ such that $v = x_i u/x_j \in G(I)$. One has $v \in J$. Since $x_i u = x_j v \in J$, one has $x_i \in J : u$, as required. Consequently, one has $q(I) \geq n-d$. Since I is Cohen–Macaulay, it follows that $h(I) \geq n-d+1$. It then turns out that, for each subset $W \subset \{x_1, \ldots, x_n\}$ with $|W| = d$, the set $\{x_1, \ldots, x_n\} \setminus W$ cannot be a vertex cover of I. Hence for each subset $W \subset \{x_1, \ldots, x_n\}$ with $|W| = d$ there is a monomial $w \in G(I)$ with $\mathrm{supp}(w) \subset W$. Since $|\mathrm{supp}(w)| \geq |\mathrm{supp}(u)| = d$, one has $\mathrm{supp}(w) = W$. Hence $\sqrt{I}$ is generated by all square-free monomials of degree d in $x_1, \ldots, x_n$. $\square$

Theorem 4.2. *A polymatroidal ideal I is Cohen–Macaulay if and only if I is*

(i) *a principal ideal,*
(ii) *a Veronese ideal, or*
(iii) *a square-free Veronese ideal.*

Proof. By using Lemma 4.1 we assume that $\sqrt{I}$ is generated by all square-free monomials of degree d in $x_1, \ldots, x_n$, where $1 \leq d \leq n$. When $d = 1$, one has $\sqrt{I} = (x_1, \ldots, x_n)$. Thus $x_1^r, \ldots, x_n^r \in G(I)$ for some $r \geq 1$. It then follows from [2, Theorem 3.4] that I is the Veronese ideal $(x_1, \ldots, x_n)^r$. When $d = n$, one has $\sqrt{I} = (x_1 \cdots x_n)$. Thus $h(I) = h(\sqrt{I}) = 1$. Hence $q(I) = 0$, and hence I must be a principal ideal. Thus in what follows we will assume that $2 \leq d < n$.

One has $h(I) = h(\sqrt{I}) = n-d+1$. Suppose that I is not square-free (or, equivalently, each monomial belonging to $G(I)$ is of degree $> d$). Let $u = \prod_{i=n-d+1}^{n} x_i^{a_i} \in G(I)$ be a monomial with $\mathrm{supp}(u) = \{x_{n-d+1}, x_{n-d+2}, \ldots, x_n\}$. For a while, we assume that $(*)$ there is a monomial $v = \prod_{i=1}^{n} x_i^{b_i} \in G(I)$ with $b_{n-d+1} > a_{n-d+1}$. Let J denote the monomial ideal generated by those monomials $w \in G(I)$ such that w is bigger than u with respect to the reverse lexicographic order induced by the ordering $x_1 > x_2 > \cdots > x_n$. As was shown in the proof of Lemma 4.1, the colon ideal $J : u$ is generated by a subset M of $\{x_1, \ldots, x_n\}$ with $\{x_1, \ldots, x_{n-d}\} \subset M$. We claim that $x_{n-d+1} \in J : u$. Using Lemma 3.1, our assumption $(*)$ guarantees that there is a variable x_j with $n-d+1 < j \leq n$ such that $u_0 = x_{n-d+1}u/x_j \in G(I)$. Since $u_0 \in J$, one has $x_{n-d+1} \in M$. Hence $q(I) \geq n-d+1$. Thus $h(I) < q(I)+1$ and I cannot be Cohen–Macaulay.

To complete our proof, we must examine our assumption $(*)$. For each d-element subset $\sigma = \{x_{i_1}, x_{i_2}, \ldots, x_{i_d}\}$ of $\{x_1, \ldots, x_n\}$, there is a monomial $u_\sigma \in G(I)$ with $\mathrm{supp}(u_\sigma) = \sigma$. If there are d-element subsets σ and τ of $\{x_1, \ldots, x_n\}$ and a variable $x_{i_0} \in \sigma \cap \tau$ with $a_{i_0} < b_{i_0}$, where a_{i_0} (resp. b_{i_0}) is the power of x_{i_0} in u_σ (resp. u_τ), then after relabelling the variables if necessary we may assume that $\sigma = \{x_{n-d+1}, x_{n-d+2}, \ldots, x_n\}$ with $i_0 = n-d+1$. In other words, the condition $(*)$ is satisfied. Thus in the case that the condition $(*)$ fails to be satisfied, there is a positive integer $e \geq 2$ such that, for each d-element subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_d}\}$ of $\{x_1, \ldots, x_n\}$, one has $u = (x_{i_1} x_{i_2} \cdots x_{i_d})^e \in G(I)$. In the argument that follows, the assumption $2 \leq d < n$ will be essential. Let $w = x_{n-d} x_{n-d+1}^{e-1} (\prod_{i=n-d+2}^{n} x_i^e) \in G(I)$. Let J denote the monomial ideal generated by those monomials $v \in G(I)$ such that v is bigger than w with respect to the reverse lexicographic order. Since $\prod_{i=n-d}^{n-1} x_i^e \in G(I)$, by using Lemma 3.1 one has $w_0 = x_{n-d}w/x_n \in J$ and $w_1 = x_{n-d+1}w/x_n \in J$. Thus the colon ideal $J : w$ is generated by a subset M of $\{x_1, \ldots, x_n\}$ with $\{x_1, \ldots, x_{n-d}, x_{n-d+1}\} \subset M$. Hence $q(I) \geq n - d + 1$, and thus we have $h(I) < q(I) + 1$, a contradiction. $\square$

As we pointed out in Section 2, a Cohen–Macaulay ideal is always unmixed. The converse is in general not true, even for matroidal ideals. For example, let $I \subset K[x_1, \ldots, x_6]$ be the monomial ideal generated by

$$x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2x_3, x_2x_4, x_2x_5, x_2x_6, x_3x_5, x_3x_6, x_4x_5, x_4x_6.$$

Then I is matroidal and unmixed. However, I is not Cohen–Macaulay.

It would, of course, be of great interest from a viewpoint of combinatorics to classify all unmixed polymatroidal ideals.

References

[1] A. Conca, J. Herzog, Castelnuovo–Mumford regularity of products of ideals, Collect. Math. 54 (2003) 137–152.

[2] J. Herzog, T. Hibi, Discrete polymatroids, J. Algebraic Combin. 16 (2002) 239–268.

[3] J. Herzog, Y. Takayama, Resolutions by mapping cones, Homology Homotopy Appl. 4 (2002) 277–294.